SIMPLE PROOFS OF GOLDBACH'S CONJECTURE

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ABSTRACT: A first theorem stating that every positive or negative even integer is the difference between two primes is proved. By applying the proof of the first theorem a simple proof of the second theorem (Goldbach's conjecture) is presented. Using the first and second theorems a lemma stating that every positive or negative odd integer is expressed by the addition and/or subtraction of three primes is proved.

1. A FIRST THEOREM

Theorem 1: Every positive or negative even integer is expressed as the difference between two primes. Or every even integer n is expressed as p - q, where p and q are primes.

Proof: Let n be some positive even integer greater than or equal to 6 which cannot be expressed by the difference between two primes. And suppose that p_i 's(i=1, 2, 3, …) are all primes from 3 to the prime p_{J+1}, where p_1=3, p_2=5, and n is greater than p_J and smaller than p_{J+1}, namely

$$p_J < n < p_{J+1}. \tag{1}$$

Then n is expressed as $2ap_x$, where a is a natural number, and p_x is a prime selected from p_1 to p_J. Here n or $2ap_x$ is expressed as the difference between a prime and a composite integer. In other words n or $2ap_x$ satisfies the following equations, where b_i's are some odd integers greater than or equal to 3 and q_i's are primes selected from p_1 to p_J:

$2ap_x + p_1 = b_1q_1$ (2-1)

$2ap_x + p_2 = b_2q_2$ (2-2)

$2ap_x + p_3 = b_3q_3$ (2-3)

………………………

$2ap_x + p_{J-1} = b_{J-1}q_{J-1}$ (2-J-1)

$2ap_x + p_J \quad = b_Jq_J$. (2-J)

It should be noted that p_x is smaller than or equal to p_J and q_i's are smaller than or equal to p_J, since $b_i \geqq 3$. We determine n or $2ap_x$ which satisfies equations (2-1) to (2-J) by using mathematical induction. Firstly let us assume J=2, then equations (2-1) to (2-J) will be

$2ap_x + p_1 = b_1q_1$ (3-1)

$2ap_x + p_2 = b_2q_2$ (3-2)

Then if we assume that p_x is $p_1$and q_1 is p_1, equation (3-1) will be

$2ap_1 + p_1 = b_1p_1$, namely (3-3)

$2a + 1 = b_1$ or $2a = b_1 - 1$. (3-4)

In this case if we further assume that q_2 is p_1 (case1: p_x is p_1, q_1 is p_1 and q_2 is p_1), equation (3-2) will be

$2ap_1 + p_2 = b_2p_1$, namely $p_2 = (b_2 - 2a)p_1$. (3-5)

Since p_2 will become a composite number by equation (3-5), case 1 does not hold true. Then if we assume that q_2 is p_2 (case2: p_x is p_1, q_1 is p_1 and q_2 is p_2), equation (3-2) will be

$2ap_1 + p_2 = b_2p_2$, namely $2ap_1 = (b_2 - 1)p_2$. (3-6)

Equation (3-6) leads to

$a = a'p_2$ and $2p_1 = (b_2\text{-}1)$ or $b_2 = 2p_1 + 1$. (3-7)

By substituting $a = a'p_2$ (a' is a natural number) in $2ap_1$, n is expressed as

$n = 2a'p_1p_2$. (3-8)

Using n defined by equation (3-8) in equations (3-1) to (3-2) we obtain

$2a'p_1p_2 + p_1 = b_1p_1$ or $b_1 = 2a'p_2 + 1$ (3-9)

$2a'p_1p_2 + p_2 = b_2p_2$ or $b_2 = 2a'p_1 + 1$. (3-10)

This means that case 2 holds true and n or $2a'p_1p_2$ satisfies equations (3-1) to (3-2).

Similarly if we assume that p_x is p_2, we obtain n defined by equation (3-8).

Secondly let us assume $p_{J\text{-}1} < n < p_J$, then to satisfy equations (2-1) to (2-J-1) n is expressed as

$n = 2a'p_1p_2p_3 \ldots p_{J\text{-}2}p_{J\text{-}1}$, (3-11)

where a' is another natural number.

Thirdly let us assume $p_J < n < p_{J+1,}$ then equation (2-J) is written as

$2a'p_1p_2p_3 \ldots p_{J\text{-}2}p_{J\text{-}1} \;+\; p_J \;=\; b_Jq_J$. (3-12)

Then if we suppose q_J is p_j (here j=1,2, …, or J-1), equation (3-12) leads to

$2a'p_1p_2p_3 \ldots p_{J\text{-}2}p_{J\text{-}1} \;+\; p_J \;=\; b_Jp_j$, or (3-13)

$p_J = p_j(b_j - 2a'p_1p_2 \ldots p_{j\text{-}1}p_{j+1} \ldots p_{J\text{-}1})$. (3-14)

This case does not hold true because p_J will become a composite number. Thus if we suppose q_J is p_J, equation (2-J) leads to

$2a'p_1p_2p_3 \ldots p_{J\text{-}2}p_{J\text{-}1} \;+\; p_J \;=\; b_Jp_J$, or (3-15)

$2a'p_1p_2p_3 \ldots p_{J\text{-}2}p_{J\text{-}1} = p_J(b_J - 1)$. (3-16)

To satisfy equation (3-16) we need following equations (a'' is another natural number)

$a' = a''p_J$, and (3-17)

$2p_1p_2p_3\ldots p_{J-2}p_{J-1} = b_J - 1$. (3-18)

By combining equations (3-17) and (3-11) n which satisfies the equations (2-1) to (2-J) can be expressed as (a is a natural number)

$n=2ap_1p_2p_3\ldots p_{J-2}p_{J-1}\, p_J$. (3-19)

However, the even number n which is defined by equation (3-19) is much greater than the above defined range ($p_J < n < p_{J+1}$). This means that there does not exist an even integer n which satisfies the equations (2-1) to (2-J) in the defined range.

In the above discussion if q_J which satisfies equation (3-12) does not exist, b_Jq_J will be some prime greater than p_J. This completes the Theorem 1.

And for n=2 or 4 the Theorem 1 holds true since 2=5-3 or 4=7-3. Thus using some primes p and q every positive even integer is written as

$n + q = p$ or $n = p - q$. (3-20)

Furthermore equation (3-20) can be written as

$-n = q - p$, (3-21)

which means that every negative even integer is expressed as the difference between two primes. This completes the Theorem 1. ☐

2. A SECOND THEOREM (GOLDBACH'S CONJECTURE)

Theorem 2: Every even integer greater than 2 is expressed as the sum of two primes. Or every even integer n greater than 2 is expressed as $p + q$, where p and q are primes.

Proof: Let n be some positive even integer greater than or equal to 6 which cannot be expressed by the sum of two primes. And suppose that p_i 's(i=1, 2, 3, …) are all primes from 3 to the prime p_{J+1}, where p_1=3, p_2=5, and n is greater than p_J and smaller than p_{J+1}, namely

$$p_J < n < p_{J+1}. \quad (4)$$

Then n is expressed as $2ap_x$, where a is a natural number, and p_x is a prime selected from p_1 to p_J. Here n or $2ap_x$ is expressed as the sum of a prime and a composite integer. In other words n or $2ap_x$ satisfies the following equations, where b_i's are some odd integers greater than or equal to 3 and q_i's are primes selected from p_1 to p_J:

$$2ap_x - p_1 = b_1q_1 \quad (5\text{-}1)$$

$$2ap_x - p_2 = b_2q_2 \quad (5\text{-}2)$$

$$2ap_x - p_3 = b_3q_3 \quad (5\text{-}3)$$

………………………

$$2ap_x - p_{J-1} = b_{J-1}q_{J-1} \quad (5\text{-}J\text{-}1)$$

$$2ap_x - p_J = b_Jq_J . \quad (5\text{-}J)$$

It should be noted that p_x is smaller than or equal to p_J and q_i's are smaller than or equal to p_J, since $b_i \geqq 3$. We determine n or $2ap_x$ which satisfies equations (5-1) to (5-J) by using mathematical induction. Firstly let us assume J=2, then equations (5-1) to (5-J) will be

$$2ap_x - p_1 = b_1q_1 \quad (6\text{-}1)$$

$$2ap_x - p_2 = b_2q_2 \quad (6\text{-}2)$$

Then if we assume that p_x is $p_1$and q_1 is p_1, equation (6-1) will be

$$2ap_1 - p_1 = b_1p_1 \text{ , namely} \quad (6\text{-}3)$$

$$2a - 1 = b_1 \text{ or } 2a = b_1 + 1. \quad (6\text{-}4)$$

In this case if we further assume that q_2 is p_1 (case1: p_x is p_1, q_1 is p_1 and q_2 is

p_1), equation (6-2) will be

$2ap_1 - p_2 = b_2p_1$, namely $p_2 = (2a - b_2)p_1$. (6-5)

Since p_2 will become a composite number by equation (6-5), case 1 does not hold true. Then if we assume that q_2 is p_2 (case2: p_x is p_1, q_1 is p_1 and q_2 is p_2), equation (6-2) will be

$2ap_1 - p_2 = b_2p_2$, namely $2ap_1 = (b_2 + 1)p_2$. (6-6)

Equation (6-6) leads to

$a = a'p_2$ and $2p_1 = (b_2+1)$ or $b_2=2p_1 - 1$. (6-7)

By substituting $a = a'p_2$ (a' is a natural number) in $2ap_1$, n is expressed as

$n=2a'p_1p_2$. (6-8)

Using n defined by equation (6-8) in equations (6-1) to (6-2) we obtain

$2a'p_1p_2 - p_1 = b_1p_1$ or $b_1=2a'p_2 - 1$ (6-9)

$2a'p_1p_2 - p_2 = b_2p_2$ or $b_2=2a'p_1 - 1$. (6-10)

This means that case 2 holds true and n or $2a'p_1p_2$ satisfies equations (6-1) to (6-2). Similarly if we assume that p_x is p_2, we obtain n defined by equation (6-8).

Secondly let us assume $p_{J-1} < n < p_J$, then to satisfy equations (5-1) to (5-J-1) n is expressed as

$n=2a'p_1p_2p_3\ldots p_{J-2}p_{J-1}$, (6-11)

where a' is another natural number.

Thirdly let us assume $p_J < n < p_{J+1},$ then equation (5-J) is written as

$2a'p_1p_2p_3\ldots p_{J-2}p_{J-1} - p_J = b_Jq_J$. (6-12)

Then if we suppose q_J is p_j (here j=1,2, …, or J-1), equation (6-12) leads to

$2a'p_1p_2p_3\ldots p_{J-2}p_{J-1} - p_J = b_Jp_j$, (6-13)

or $p_J = p_j(- b_j + 2a'p_1p_2\ldots p_{j-1}p_{j+1}\ldots p_{J-1})$. (6-14)

This case does not hold true because p_J will become a composite number. Thus if we suppose q_J is p_J, equation (5-J) leads to

$$2a'p_1p_2p_3\ldots p_{J-2}p_{J-1} - p_J = b_Jp_J\,, \qquad (6\text{-}15)$$

or $$2a'p_1p_2p_3\ldots p_{J-2}p_{J-1} = p_J(b_J+1)\,. \qquad (6\text{-}16)$$

To satisfy equation (6-16) we need following equations (a'' is another natural number)

$$a' = a''p_J, \quad \text{and} \qquad (6\text{-}17)$$

$$2p_1p_2p_3\ldots p_{J-2}p_{J-1} = b_J+1\,. \qquad (6\text{-}18)$$

By combining equations (6-17) and (6-11), n which satisfies the equations (5-1) to (5-J) can be expressed as (a is a natural number)

$$n=2ap_1p_2p_3\ldots p_{J-2}p_{J-1}\,p_J. \qquad (6\text{-}19)$$

However, the even number n which was defined by equation (6-19) is much greater than the above defined range ($p_J < n < p_{J+1}$). This means that there does not exist an even integer n which satisfies the equations (5-1) to (5-J) in the defined range.

In the above discussion if q_J which satisfies equation (6-12) does not exist, b_Jq_J will be some prime greater than p_J. This completes the Theorem 2.

And for n= 4 or 6 the Theorem 2 holds true since 4=2+2 or 6=3+3. Thus using some primes p and q every positive even integer greater than 2 is written as

$$n - q = p \text{ or } n = p + q. \qquad (6\text{-}20)$$

This completes the Theorem 2. □

3. A LEMMA

Lemma 1: Every positive or negative odd integer is expressed as the addition and/or subtraction of three primes. Or every positive or negative odd integer m is expressed as

$p+q+r$ or $p-q+r$ and so on, where p, q, and r are primes.

Proof: Let m be a positive odd integer. Using some even integer n and a prime r, integer m is written as

$$m = n + r \tag{7-1}$$

Here by the theorems 2 and 1 the even integer n is expressed as

$$n= p +q, \text{ or} \tag{7-2}$$

$$n= p - q , \tag{7-3}$$

where p and q are primes. Using equations (7-1) and (7-2), m is expressed as

$$m= p + q+ r . \tag{7-4}$$

Similarly using equations (7-1) and (7-3), m is expressed as

$$m= p - q+ r \tag{7-5}$$

Furthermore equation (7-5) can be written as

$$-m=q- p- r . \tag{7-6}$$

This means every negative odd integer is expressed as the addition and/or subtraction of three primes.

This completes the Lemma 1. □

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