A GENERALIZED FIBONACCI-LIKE SEQUENCE

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ABSTRACT: A generalized Fibonacci-like sequence $\{S_n\}$ and its characteristic equation are presented. Fibonacci, Lucas, Pell, and Padovan sequences are included in the sequence $\{S_n\}$. It is shown that the equation has only one real solution x_0 in the range greater than 1 and the ratio S_n/S_{n-1}^m (m: a natural number) converges on x_0 under certain conditions.

1. DEFINITIONS AND A LEMMA

Let k be any integer greater than or equal to 2, and a_1 and a_k be natural numbers and other a_i (i=2,..,k-1) non-negative integers. Then, using natural numbers b_i (i=1,2,..,k), a generalized Fibonacci-like sequence $\{S_n\}$ is defined by

S_i(i=0,1,...,k-2) any non-negative integers, S_{k-1} any natural number,

and

$$S_n = a_1 S_{n-1}^{b_1} + a_2 S_{n-2}^{b_2} + \dots + a_k S_{n-k}^{b_k}, \quad for\ n \geq k. \tag{1}$$

Here, we assume that as n increases the ratio $S_n/(S_{n-1})^m$ converges on a real number x, where m is a natural number. To determine x, let us write

$$\frac{S_n}{(S_{n-1})^m} = \frac{S_{n-1}}{(S_{n-2})^m} = \frac{S_{n-2}}{(S_{n-3})^m} = \ldots = \frac{S_{n-k+1}}{(S_{n-k})^m} = x. \qquad (2)$$

Relations (2) can be rewritten as

$$S_{n-k+1} = x\, S_{n-k}^{m}, \qquad (3\text{-}1)$$

$$S_{n-k+2} = x\ S_{n-k+1}^{m} = x^{m+1} S_{n-k}^{m^2}, \qquad (3\text{-}2)$$

$$S_{n-k+3} = x\ S_{n-k+2}^{m} = x^{m^2+m+1} S_{n-k}^{m^3}, \qquad (3\text{-}3)$$

$$S_n = x\ S_{n-1}^{m} = x^{m^{k-1}+m^{k-2}+\ldots+m+1} S_{n-k}^{m^k}. \qquad (3\text{-k})$$

By putting relations (3-1) to (3-*k*) in (1), we obtain

$$x^{m^{k-1}+m^{k-2}+\ldots+m+1} S_{n-k}^{m^k} = \sum_{i=1}^{k-1} a_i\, x^{(m^{k-i-1}+m^{k-i-2}+\ldots+m+1)b_i} S_{n-k}^{m^{k-i}b_i} + a_k S_{n-k}^{b_k}. \qquad (4)$$

To have the identity (4) hold for any S_{n-k}, we need the following relations

$$S_{n-k}^{m^k} = S_{n-k}^{m^{k-i}b_i} = S_{n-k}^{b_k} \quad (i=1,2,\ldots,k-1) \qquad (5)$$

From (5) we obtain $b_1=m$, $b_2=m^2$,..., $b_k=m^k$:

$$b_i=m^i \qquad (i=1,2,\ldots,k). \qquad (6)$$

Putting (6) in (1) and (4) yields

$$S_n = a_1 S_{n-1}^{m} + a_2 S_{n-2}^{m^2} + \ldots + a_k S_{n-k}^{m^k}, \quad \textit{for } n \geq k, \qquad (7)$$

and a characteristic equation

$$x^{m^{k-1}+m^{k-2}+\dots+m+1} = a_1 x^{m^{k-1}+m^{k-2}+\dots+m} + a_2 x^{m^{k-1}+m^{k-2}+\dots+m^2} + \dots + a_{k-1} x^{m^{k-1}} + a_k. \qquad (8)$$

Suppose that m=1, then the characteristic equation (8) of (7) leads to

$$x^k = a_1x^{k-1} + a_2x^{k-2} + \dots + a_{k-1}x + a_k. \qquad (9)$$

Suppose that m=2 and k=3, then the sequence (7) and the characteristic equation (8) are as follows:

$$S_n = a_1S_{n-1}^{\ 2} + a_2S_{n-2}^{\ 4} + a_3S_{n-3}^{\ 8}, \quad \text{for } n \geqq 3, \qquad (10)$$

and

$$x^7 = a_1x^6 + a_2x^4 + a_3. \qquad (11)$$

Remark 1: It is easy to verify that equation (8) has only one real solution in the range x>1. Namely, since a_1 and a_k are natural numbers and other a_i's(i=2,3,...,k-1) non-negative integers, by dividing both sides of (8) by

$$x^{m^{k-1}+m^{k-2}+\dots+m},$$

we obtain

$$x = \mathrm{f(x)} = a_1 + \frac{a_2}{x^m} + \frac{a_3}{x^{m^2+m}} + \dots + \frac{a_k}{x^{m^{k-1}+m^{k-2}+\dots+m}}. \qquad (12)$$

In the range x>1, as x increases from 1 whereas function $f(x)$ decreases from (a_1+a_2+...+a_k) ($\geqq$ 2) to a1 monotonously, equation (12) has only one real solution x_0 and so does equation (8).

Remark 2: It is obvious from equation (12) that

$$1 \leqq a_1 < x_0 < (a_1+a_2+...+a_k) \quad , \tag{13}$$

and

$$\text{if } x < x_0, \text{ then } f(x) > x_0, \text{ and if } x > x_0, \text{ then } f(x) < x_0, \tag{14}$$

where $f(x)$ is defined by (10) in the range $x>1$.

Further, it should be noted that x_0 is also the solution of the following equation:

$$x = f(f(x)) \; . \tag{15}$$

Definition 1: If equation (15) has no real solution other than x_0 in the range $x>1$, then we assume that the function $f(x)$ defined by (12) is "simple."

Suppose that function $f(x)$ is NOT simple. Then, there exist real numbers e, g ($e \neq x_0$, $g \neq x_0$, $e \neq g$, $1<e$, $1<g$) such that $e=f(g)$ and $g=f(e)$. In this case, since $e=f(g)=f(f(e))$ and $g=f(e)=f(f(g))$, equation (15) has at least two real solutions other than x_0 in the range $x>0$.

For example, in the cases of Fibonacci numbers (m=1, k=2, a_1=a_2=1, S_0=0, S_1=1) [1], Lucas numbers (m=1, k=2, a_1=a_2=1, S_0=2, S_1=1) [3], and Pell numbers (m=1, k=2, a_1=2, a_2=1, S_0=0, S_1=1) [2], function $f(x)$ is expressed as a_1+a_2/x, which is simple.

Lemma 1: Let function $f(x)$ be simple. Then if $1 < x < x_0$, then $x < f(f(x))$, and if $x > x_0$, then $x > f(f(x))$, where x_0 is the only one solution of equation (10) in the range $x>1$.

Proof: Let us write

$$F(x) = x - f(f(x)). \qquad (16)$$

When x increases from 1 to infinity, as function $f(x)$ decreases from $(a_1+a_2+...+a_k)$ to a_1 monotonously, function $f(f(x))$ increases from $f(a_1+a_2+...+a_k)$ (>1) to $f(a_1)$ monotonously. Thus, $F(x)<0$ at $x=1$, and $F(x)>0$ at infinity. Further, $F(x_0)=0$, for $f(f(x_0))=f(x_0)=x_0$.

Then, since function $f(x)$ is simple, $F(x)$ never reaches 0 other than x_0, we see that if $1< x< x_0$, then $F(x)<0$, and if $x>x_0$, then $F(x)>0$, establishing the Lemma 1.

2. A THEOREM AND THE PROOF

Theorem 1: In a generalized Fibonacci-like sequence $\{S_n\}$ defined by (7), suppose that function $f(x)$ defined by (12) is simple. Then, as n increases the ratio $S_n/S_{n-1}{}^m$ converges on x_0, where x_0 is the only one real solution of equation (12) or (8) in the range $x>1$.

Proof: Let us write

$$\frac{S_{n-i}}{(S_{n-i-1})^m} = \frac{S_{n-i}}{S_{n-i-1}^{\ m}} = y_{n-i}, \quad for \ \ i = 0, 1, \cdots, k-1. \qquad (21)$$

Relations (21) can be rewritten as

$$S_{n-(k-1)} = y_{n-(k-1)}\, S_{n-k}^{\ m}\ , \qquad (22\text{-}1)$$

$$S_{n-(k-2)} = y_{n-(k-2)}\, S_{n-(k-1)}^{\ m} = y_{n-(k-2)}\, y_{n-(k-1)}^{\ m}\, S_{n-k}^{\ m^2}, \qquad (22\text{-}2)$$

$$S_{n-(k-3)} = y_{n-(k-3)}\, S_{n-(k-2)}^{\ m} = y_{n-(k-3)}\, y_{n-(k-2)}^{\ m}\, y_{n-(k-1)}^{\ m^2}\, S_{n-k}^{\ m^3}, \qquad (22\text{-}3)$$

$$S_n \quad = y_n\, S_{n-1}^{\ m} = y_n\, y_{n-1}^{\ m} \cdots y_{n-(k-2)}^{\ m^{k-2}}\, y_{n-(k-1)}^{\ m^{k-1}}\, S_{n-k}^{\ m^k}. \qquad (22\text{-}k)$$

By putting relations (22-1) to (22-k) in (7), we obtain

$$y_n\, y_{n-1}^{\,m} \cdots y_{n-(k-1)}^{\,m^{k-1}}\, S_{n-k}^{\,m^k} = \sum_{i=1}^{k-1} a_i\, y_{n-i}^{\,m^i}\, y_{n-(i+1)}^{\,m^{i+1}} \cdots y_{n-(k-1)}^{\,m^{k-1}}\, S_{n-k}^{\,m^k} + a_k\, S_{n-k}^{\,m^k}. \qquad (23)$$

Dividing both sides of (23) by

$$y_{n-1}^{\,m}\, y_{n-2}^{\,m^2} \cdots y_{n-(k-1)}^{\,m^{k-1}}\; S_{n-k}^{\,m^k}$$

leads to

$$y_n = a_1 + \frac{a_2}{y_{n-1}^{\,m}} + \frac{a_3}{y_{n-1}^{\,m}\; y_{n-2}^{\,m^2}} + \cdots + \frac{a_k}{y_{n-1}^{\,m}\, y_{n-2}^{\,m^2} \cdots y_{n-(k-1)}^{\,m^{k-1}}}, \qquad (24)$$

which is equivalent to definition (7) of the sequence $\{S_n\}$. It should be noted that by replacing each y_i ($i=n-1,n-2,\ldots,n-(k-1)$) by x on the right side of (24) we obtain the same function as the function $f(x)$ defined by (12).

First, since a_1 and S_{k-1} are natural numbers in (7), if $n \geqq k-1$, then $S_n \geqq 1$. Observe that a_k is also a natural number. Hence, if $n \geqq 2k-1$, then

$$\frac{S_n}{S_{n-1}^{\,m}} = y_n > a_1 \geq 1. \qquad (25)$$

Second, considering (25) and (24), we see that if $n \geqq 3k-1$, then

$$1 < y_n < (a_1+a_2+\ldots+a_k). \qquad (26)$$

Comparing (11) and (26), we can choose two real numbers c_1 and d_1 such that

$$1 < c_1 < x_0 < d_1 < (a_1+a_2+\ldots+a_k), \text{ and } c_1 < y_n < d_1. \qquad (27)$$

The second inequality of (27) leads to

$$\frac{1}{d_1} < \frac{1}{y_n} < \frac{1}{c_1}. \qquad (28)$$

By putting inequality (28) in (24) and using function $f(x)$ defined by (12), we see that if

$n \geqq 4k$-1, then

$$1 < f(d_1) < y_n < f(c_1). \tag{29}$$

Considering the first inequality1 of (27) and relation (14), we see that

$$f(d_1) < x_0 < f(c_1). \tag{30}$$

Then, inequality of (29) leads to

$$\frac{1}{f(c_1)} < \frac{1}{y_n} < \frac{1}{f(d_1)} . \tag{31}$$

By putting inequality (31) in (24) and using function $f(x)$ defined by (12), we see that if $n \geqq 5k$-1, then

$$1 < f(f(c_1)) < y_n < f(f(d_1)) . \tag{32}$$

Observe that function $f(x)$ is simple. Hence, it follows from Lemma 1 and (27), (30), (32) that

$$c_1 < f(f(c_1)) < y_n < f(d_1) < d_1 \text{ , and } f(f(c_1)) < x_0 < f(f(d_1)). \tag{33}$$

Let us write

$$f(f(c_1))= c_2, \text{ and } f(f(d_1))= d_2 \quad , \tag{34}$$

then inequalities (34) can be rewritten as

$$1< c_1 < c_2 < y_n < d_2 < d_1, \text{ and } c_2 < x_0 < d_2 \quad . \tag{35}$$

Further, if we start the above procedure for $n \geqq 4k$-1 with inequalities (27), then inequalities (35) holds for $n \geqq 6k$-1.

Hence, as n increases, there exist sequences $\{c_i\}$ and $\{d_i\}$ such that

$$1< c_1 < c_2 < ... < c_i < y_n < d_i < ...< d_2 < d_1, \text{ and } c_i < x_0 < d_i \quad , \tag{36}$$

where $c_i=f(f(c_{i-1}))$ and $d_i=f(f(d_{i-1}))$. This completes the Theorem 1. □

3. A MORE GENERALIZED CASE (a_1=0)(this part needs to be revised)

Suppose that $a_1=0$ and $a_p \geqq 1$ ($1<p<k$, $k \geqq 3$) in a generalized sequence $\{S_n\}$ defined by (7) to include Padovan sequence ($m=1$, $k=3$, $a_1=0$, $a_2=a_3=1$, $S_0=S_1=S_2=1$) ［4］. In this case it is easy to verify that function $f(x)$ defined by (12) ($a_1=0$, $a_p \geqq 1$, $a_k \geqq 1$) is simple in the sense of Definition 1.

Hence, considering the above proof, if there exists a number n_0 such that ratio $S_n/S_{n-1}{}^m > 1$ for $n>n_0$, then the ratio $S_n/S_{n-1}{}^m$ converges on x_0 as n increases, where x_0 is the only one real solution of equation (12) or (8) in the range $x>1$.

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REFERENCES

1. G. E. Andrews. "Some Formulae for the Fibonacci Sequence with Generalizations." The Fibonacci Quarterly 7.2 (1969): 113-30.
2. W. L. McDaniel. "Triangular Numbers in the Pell Sequence."" The Fibonacci Quarterly 34 (1996): 105-07.
3. N. Robbins. "A New Formula for Lucas Numbers." The Fibonacci Quarterly 29.4 (1991): 362-63.
4. I. Stewart. "Tales of a Neglected Number." Scientific American 274 (1996): 102-03.

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