A generalization of Fibonacci-like sequences

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Abstract

A generalized Fibonacci-like sequence $\{S_n\}$(for $n \geq k$) is defined by $a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k}$. By substituting x for $\frac{S_{n-i}}{S_{n-1-i}}(i = 0, 1, \ldots, k-1)$ in the sequence, an equation for determining x is derived in the form of $x = h(x)$. The equation has only one real positive solution x_0. It is proved that the ratio $\frac{S_n}{S_{n-1}}$ converges to x_0 regardless of the values of $S_{k-1}, \ldots, S_0$ under the condition that the function $h(x)$ has certain properties. It is shown that Fibonacci, Lucas, and Pell sequences satisfy the condition.

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1 Definitions and Lemmas

Let k be any integer ≥ 2, and a_1 and a_k be natural numbers. And suppose that other $a_i(i = 2, \ldots, k-1)$ are non-negative integers.

Then, a generalized Fibonacci-like sequence $\{S_n\}$ is defined by

$$S_n = a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k}, \quad \text{for } n \geq k, \tag{1.1}$$

where $S_i(i = 0, 1, \ldots, k-2)$ are any non-negative integers, and S_{k-1} is any natural number. The sequence $\{S_n\}$ includes Fibonacci numbers [1], Lucas numbers [3], and Pell numbers [2].

Here, we assume that as n increases the ratio $\frac{S_n}{S_{n-1}}$ converges to a real number x. To determine x, let us write

$$\frac{S_n}{S_{n-1}} = \frac{S_{n-1}}{S_{n-2}} = \frac{S_{n-2}}{S_{n-3}} = \cdots = \frac{S_{n-(k-1)}}{S_{n-k}} = x. \tag{1.2}$$

Relations (1.2) can be rewritten as

$$S_{n-(k-1)} = x S_{n-k}, \tag{1.3}$$

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$$S_{n-(k-2)} = xS_{n-(k-1)} = x^2 S_{n-k}, \tag{1.3-1}$$

$$S_{n-(k-3)} = xS_{n-(k-2)} = x^3 S_{n-k}, \tag{1.3-2}$$

and finally

$$S_n = xS_{n-1} = x^k S_{n-k}. \tag{1.3-(k-1)}$$

By putting relations (1.3) to (1.3-(k-1)) in (1.1), we obtain

$$x^k S_{n-k} = \sum_{i=1}^{k-1} a_i x^{k-i} S_{n-k} + a_k S_{n-k}. \tag{1.4}$$

Dividing both sides of (1.4) by S_{n-k} yields an equation

$$x^k = a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_{k-1} x + a_k. \tag{1.5}$$

Lemma 1.1. *Equation* (1.5) *has only one real solution in the range* $x > 0$.

Proof. We divide the both sides of (1.5) by x^{k-1}. Hence we obtain

$$x = a_1 + \frac{a_2}{x} + \frac{a_3}{x^2} + \cdots + \frac{a_k}{x^{k-1}}, \tag{1.6}$$

where a_1 and a_k are natural numbers and other $a_i (i = 2, 3, \ldots, k-1)$ are non-negative integers. In the range $x > 0$, since the right side of (1.6) decreases strictly from infinity to a_1 as x increases from 0, equation (1.6) has only one real positive solution and so does equation (1.5). □

Definition 1.2. We define the right-side function of (1.6) as a function $h(x)$, namely

$$h(x) = a_1 + \frac{a_2}{x} + \frac{a_3}{x^2} + \cdots + \frac{a_k}{x^{k-1}}. \tag{1.7}$$

Using the function $h(x)$, equation (1.6) can be rewritten as

$$x = h(x). \tag{1.8}$$

Definition 1.3. We define the only one real positive solution to equation (1.5) or (1.8) as x_0, namely

$$x_0 = h(x_0). \tag{1.9}$$

In the case of Fibonacci and Lucas sequences ($k = 2, a_1 = a_2 = 1$), equation (1.5) is reduced to

$$x^2 - x - 1 = 0,$$

where the real positive solution x_0 is the golden ratio: $(1 + \sqrt{5})/2$.

It is obvious from equation (1.6) that

$$1 \leq a_1 < x_0 < (a_1 + a_2 + \cdots + a_k). \tag{1.10}$$

Here $(a_1 + a_2 + \cdots + a_k) \geq (a_1 + 1)$, for at least a_1 and a_k are natural numbers.

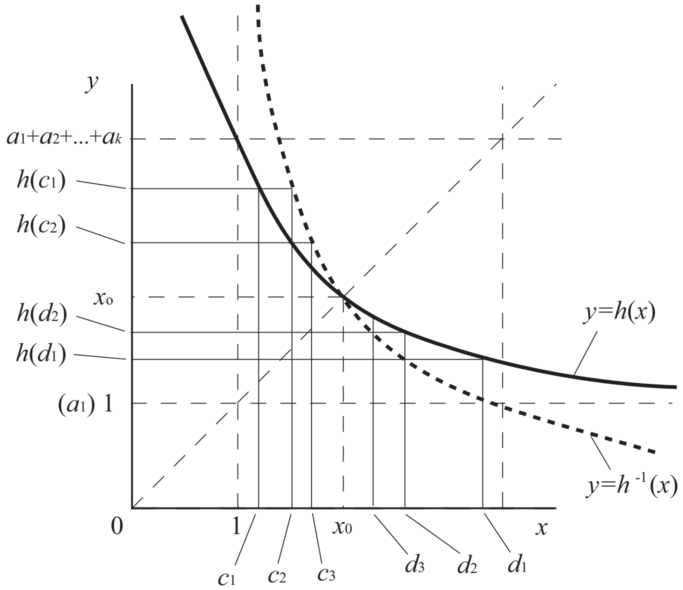

Figure 1: An example of a simple function $h(x)$ and its inverse function $h^{-1}(x)$

Remark 1.4. Since the function $h(x)$ is a decreasing function,

$$\text{if } 0 < x < x_0, \text{ then } x_0 < h(x), \text{ and if } x_0 < x, \text{ then } h(x) < x_0.$$

We then introduce a function $h(h(x))$, a function of the function $h(x)$. We then put that $c_{j+1} = h(h(c_j))$ and $d_{j+1} = h(h(d_j))$, where $0 < c_j < x_0$, $x_0 < d_j$, and j is any positive integer. Since the function $h(h(x))$ is an increasing function, we obtain the following remark.

Remark 1.5. If $c_j < h(d_j)$ and $h(c_j) < d_j$, then

$$c_{j+1} < h(d_{j+1}) \text{ and } h(c_{j+1}) < d_{j+1}.$$

Further, note that x_0 is also one of the solutions to the following equation:

$$x = h(h(x)). \tag{1.11}$$

Definition 1.6. If equation (1.11) has no real positive solution other than x_0, then we suppose that the function $h(x)$ defined by (1.7) is "simple".

Let the function $h(x)$ be *NOT* simple. Then, there exists a real positive number $e(\neq x_0)$ such that $e = h(h(e))$. In this case, if we put $g = h(e) \neq x_0$, then $e = h(g)$ and $g = h(e) = h(h(g))$. Hence, equation (1.11) has at least two real positive solutions e, g other than x_0.

In this case, using the inverse function $h^{-1}(x)$ of the function $h(x)$, the relations $g = h(e)$ and $e = h(g)$ can be rewritten as

$$g = h^{-1}(e) = h(e) \quad \text{and } e = h^{-1}(g) = h(g). \tag{1.12}$$

This means that if the function $h(x)$ is simple, $h(x)$ and $h^{-1}(x)$ never cross at any point other than x_0 in the range $x > 0$ as shown in Figure 1.

We then use the above-mentioned definitions $c_{j+1} = h(h(c_j))$ and $d_{j+1} = h(h(d_j))$, where $0 < c_j < x_0$ and $x_0 < d_j$. Referring to Figure 1, under the condition that the function $h(x)$ is simple, if $x < x_0$, then $h(x) < h^{-1}(x)$, and if $x_0 < x$, then $h^{-1}(x) < h(x)$. Hence, since $h(c_j) = h^{-1}(c_{j+1})$, we have $c_j < c_{j+1} < x_0$. Similarly, since $h(d_j) = h^{-1}(d_{j+1})$, we have $x_0 < d_{j+1} < d_j$. Thus, as j increases, the numbers c_j and d_j converge to x_0. These are proved as follows:

Lemma 1.7. *Let the function $h(x)$ be simple. And suppose that $c_{j+1} = h(h(c_j))$ and $d_{j+1} = h(h(d_j))$, where $0 < c_j < x_0$ and $x_0 < d_j$. Then*

$$c_j < c_{j+1} < x_0 \quad \textit{and} \quad x_0 < d_{j+1} < d_j.$$

Proof. First, from Remark 1.4, we have $x_0 < h(c_i)$. Then by applying Remark 1.4 once more, we have $h(h(c_j)) = c_{j+1} < x_0$. In the same way, we have $h(d_j) < x_0$, which leads to $x_0 < h(h(d_j)) = d_{j+1}$. Second, let us define that

$$H(x) = x - h(h(x)). \tag{1.13}$$

When x increases from 0 to infinity, as the function $h(x)$ decreases from infinity to $a_1(\geq 1)$, the function $h(h(x))$ increases from a_1 to $h(a_1)$. Thus, $H(x) < 0$ when $0 < x \leq 1$, and $H(x) > 0$ at infinity. Further, $H(x_0) = 0$, for $h(h(x_0)) = h(x_0) = x_0$.

Then, since $h(x)$ is simple, $H(x)$ never reaches 0 other than x_0. Hence, we see that if $0 < x < x_0$, then $H(x) < 0$, and if $x_0 < x$, then $H(x) > 0$. This means that $c_j < c_{j+1}$ and $d_{j+1} < d_j$, establishing Lemma 1.7. □

Lemma 1.8. *We assume the same conditions as in Lemma 1.7 and that $0 < c_1 < x_0$ and $x_0 < d_1$. Then, as j increases, the numbers c_j and d_j converge to x_0.*

Proof. It follows from Lemma 1.7 that

$$c_1 < c_2 < \cdots < c_{j-1} < c_j < x_0 < d_j < d_{j-1} < \cdots < d_2 < d_1. \tag{1.14}$$

Thus, by the Weierstrass' theorem the sequence $\{c_j\}$ converges to $\sup\{c_j\}$. Suppose that $\sup\{c_j\} = c_s < x_0$, then there exists a number c such that $c < c_s < h(h(c)) < x_0$. Then if we put that $0 < \varepsilon < c_s - c$, then $c < c_s - \varepsilon$ and there is an integer j_0, which satisfies the following relation:

$$c < c_s - \varepsilon < c_{j_0} < c_s \quad \text{and } c_s < h(h(c)) < h(h(c_{j_0})) = c_{j_0+1} < x_0. \tag{1.15}$$

This means that c_s is not $\sup\{c_j\}$ and x_0 is $\sup\{c_j\}$. Thus, c_j converges to x_0. On the contrary, the sequence $\{d_j\}$ converges to $\inf\{d_j\}$, which is x_0. □

If equation (1.11) has one pair of real positive solutions other than x_0 as shown in Figure 2, the function $h(x)$ is not simple. From Figure 2, we see that $c_{j+1} = h(h(c_j)) < c_j < x_0$, and $x_0 < d_j < h(h(d_j)) = d_{j+1}$. Thus, as j increases, c_j and d_j move away from x_0.

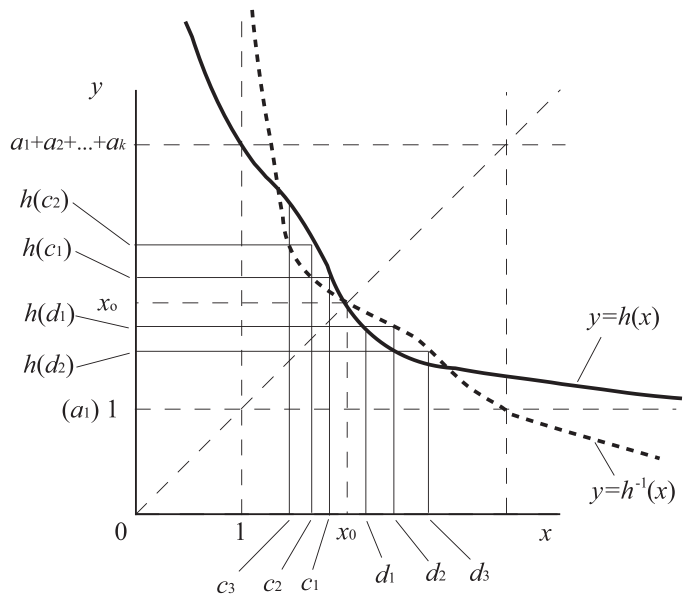

Figure 2: An example of a non-simple function $h(x)$ and its inverse function $h^{-1}(x)$

2 A Theorem and the proof

Theorem 2.1. *In a generalized Fibonacci-like sequence* $\{S_n\}$ *defined by* (1.1)*, suppose that the function* $h(x)$ *defined by* (1.7) *is simple in the sense of Definition* 1.6*. Then, as* n *increases the ratio* $\frac{S_n}{S_{n-1}}$ *converges to* x_0*, where* x_0 *is the only one real positive solution to equation* (1.8) *or* (1.5)*.*

Proof. First, since a_1 and S_{k-1} are natural numbers in (1.1), if $n \geq k$, then $S_n \geq 1$. Further since a_k is also a natural number, if $n \geq 2k$, then

$$S_n \geq a_1 S_{n-1} + a_k S_{n-k} > a_1 S_{n-1}. \tag{2.1}$$

From (2.1) it follows that

$$\frac{S_n}{S_{n-1}} > a_1 \geq 1, \quad \text{for } n \geq 2k. \tag{2.2}$$

Here, suppose that $n \geq 3k$ and let us write

$$\frac{S_{n-i}}{S_{n-(i+1)}} = y_{n-i} > a_1, \quad \text{for } i = 0, 1, \ldots, k-1. \tag{2.3}$$

Relations (2.3) can be rewritten as

$$S_{n-(k-1)} = y_{n-(k-1)} S_{n-k}, \tag{2.4}$$

$$S_{n-(k-2)} = y_{n-(k-2)}S_{n-(k-1)} = y_{n-(k-2)}y_{n-(k-1)}S_{n-k}, \tag{2.4-1}$$

$$S_{n-(k-3)} = y_{n-(k-3)}S_{n-(k-2)} = y_{n-(k-3)}y_{n-(k-2)}y_{n-(k-1)}S_{n-k}, \tag{2.4-2}$$

and finally

$$S_n = y_nS_{n-1} = y_ny_{n-1}y_{n-2}\cdots y_{n-(k-2)}y_{n-(k-1)}S_{n-k}. \tag{2.4-(k-1)}$$

By putting relations (2.4) to (2.4-(k-1)) in (1.1), we obtain

$$y_ny_{n-1}\cdots y_{n-(k-1)}S_{n-k} = \sum_{i=1}^{k-1} a_iy_{n-i}y_{n-(i+1)}\cdots y_{n-(k-1)}S_{n-k} + a_kS_{n-k}. \tag{2.5}$$

Dividing both sides of (2.5) by

$$y_{n-1}y_{n-2}\cdots y_{n-(k-1)}S_{n-k}$$

leads to

$$y_n = a_1 + \frac{a_2}{y_{n-1}} + \frac{a_3}{y_{n-1}y_{n-2}} + \cdots + \frac{a_k}{y_{n-1}y_{n-2}\cdots y_{n-(k-1)}}, \tag{2.6}$$

which is equivalent to the sequence $\{S_n\}$ defined by (1.1). It should be noted that by replacing each $y_i(i = n-1, n-2, \ldots, n-(k-1))$ by x in the right side of (2.6) we obtain the function $h(x)$ defined by (1.7).

Second, considering (2.2) and (2.6), we see that

$$1 \leq a_1 < y_n < (a_1 + a_2 + \cdots + a_k), \quad \text{for } n \geq 3k. \tag{2.7}$$

Comparing (1.10) and (2.7), we can choose two real numbers c_1 and d_1 such that

$$1 < c_1 < y_n < d_1 \quad \text{for } n = 3k, 3k+1, \ldots, 4k-1, \tag{2.8}$$

$$1 \leq a_1 < c_1 < x_0 < d_1 < (a_1 + a_2 + \cdots + a_k). \tag{2.9}$$

The inequality (2.8) leads to

$$\frac{1}{d_1} < \frac{1}{y_n} < \frac{1}{c_1} \quad \text{for } n = 3k, 3k+1, \ldots, 4k-1. \tag{2.10}$$

By putting inequality (2.10) in (2.6) we see that

$$a_1 + \frac{a_2}{d_1} + \frac{a_3}{d_1^2} + \cdots + \frac{a_k}{d_1^{k-1}} < y_n < a_1 + \frac{a_2}{c_1} + \frac{a_3}{c_1^2} + \cdots + \frac{a_k}{c_1^{k-1}} \quad \text{for } n = 4k. \tag{2.11}$$

Using the function $h(x)$ defined by (1.7), inequality (2.11) can be expressed as

$$h(d_1) < y_n < h(c_1) \quad \text{for } n = 4k. \tag{2.12}$$

Here, by substituting d_1 and c_1 for x in (1.7), respectively, it follows that

$$a_1 < h(d_1) \quad \text{and} \quad h(c_1) < (a_1 + a_2 + \cdots + a_k).$$

We then put that $c_{j+1} = h(h(c_j))$ and $d_{j+1} = h(h(d_j))$, where j is any positive integer. By comparing (2.8) and (2.12), the numbers c_1 and d_1 are classified into three cases:
Case 1 where $c_1 < h(d_1)$ and $h(c_1) < d_1$,
Case 2 where $h(d_1) < c_1$ and $h(c_1) < d_1$, and
Case 3 where $c_1 < h(d_1)$ and $d_1 < h(c_1)$.

It should be noted that there is no case where $h(d_1) < c_1$ and $d_1 < h(c_1)$. Because if $h(d_1) < c_1$, then $h(c_1) < h(h(d_1))$. Considering Lemma 1.7, we see that $h(h(d_1)) < d_1$ and hence $h(c_1) < d_1$.

(*Case* 1 where $c_1 < h(d_1)$ and $h(c_1) < d_1$)

In this case, from Remark 1.5, $c_j < h(d_j)$ and $h(c_j) < d_j$ for any positive integer j. And since $h(x)$ is a decreasing function, it follows that

$$d_{j+1} = h(h(d_j)) < h(c_j) \text{ and } h(d_j) < h(h(c_j)) = c_{j+1}. \tag{2.13}$$

Then, inequality (2.8), combined with inequality (2.12), leads to

$$c_1 < h(d_1) < y_n < h(c_1) < d_1 \quad \text{for } n = 4k, 4k+1, \ldots, 5k-1. \tag{2.14}$$

Inequality (2.14) leads to

$$\frac{1}{h(c_1)} < \frac{1}{y_n} < \frac{1}{h(d_1)} \quad \text{for } n = 4k, 4k+1, \ldots, 5k-1. \tag{2.15}$$

By putting inequality (2.15) in (2.6) and using the function $h(x)$, we see that

$$c_2 = h(h(c_1)) < y_n < h(h(d_1)) = d_2 \quad \text{for } n = 5k. \tag{2.16}$$

We then apply Lemma 1.7 and inequality (2.13) to (2.14) and (2.16). This leads to

$$c_1 < h(d_1) < c_2 < y_n < d_2 < h(c_1) < d_1 \quad \text{for } n = 5k, \ldots, 6k-1. \tag{2.17}$$

By repeating the above-mentioned process, when $n \geq (2j+1)k$, the number y_n is expressed as

$$c_1 < c_2 < \cdots < c_j < y_n < d_j < \cdots < d_2 < d_1. \tag{2.18}$$

According to Lemma 1.8, as j increases, the number c_j and d_j converge to x_0 in (2.18), and so does y_n.

(*Case* 2 where $h(d_1) < c_1$ and $h(c_1) < d_1$)

In this case, from Lemma 1.7, $d_{j+1} < d_j$. And considering that $h(x)$ is a decreasing function,we see

$$h(d_j) < h(d_{j+1}) \tag{2.19}$$

Then, inequality (2.12) can be expressed as

$$h(d_1) < y_n < h(c_1) < d_1 \quad \text{for } n = 4k. \tag{2.20}$$

Since $h(d_1) < c_1$, inequalities (2.8) and (2.20) can be unified into

$$h(d_1) < y_n < d_1 \quad \text{for } n = 3k+1, 3k+2, \ldots, 4k. \tag{2.21}$$

Inequality (2.21) leads to

$$\frac{1}{d_1} < \frac{1}{y_n} < \frac{1}{h(d_1)} \quad \text{for } n = 3k+1, 3k+2, \ldots, 4k. \tag{2.22}$$

By putting inequality (2.22) in (2.6) and using the function $h(x)$, we see that

$$h(d_1) < y_n < h(h(d_1)) = d_2 \quad \text{for } n = 4k+1. \tag{2.23}$$

Applying Lemma 1.7 to (2.21) and (2.23) yields

$$h(d_1) < y_n < d_2 < d_1 \quad \text{for } n = 4k+1, 4k+2, \ldots, 5k. \tag{2.24}$$

Inequality (2.24) leads to

$$\frac{1}{d_2} < \frac{1}{y_n} < \frac{1}{h(d_1)} \quad \text{for } n = 4k+1, \ldots, 5k \tag{2.25}$$

By putting inequality (2.25) in (2.6) and using the function $h(x)$, we see that

$$h(d_2) < y_n < h(h(d_1)) = d_2 \quad \text{for } n = 5k+1. \tag{2.26}$$

Considering inequality (2.19) and Lemma 1.7, inequalities (2.24) and (2.26) lead to

$$h(d_1) < h(d_2) < y_n < d_2 < d_1 \quad \text{for } n = 5k+1, 5k+2, \ldots, 6k. \tag{2.27}$$

By repeating the above-mentioned process, when $n \geq ((2j+1)k+1)$, the number y_n is expressed as

$$h(d_1) < h(d_2) < \cdots < h(d_j) < y_n < d_j < \cdots < d_2 < d_1. \tag{2.28}$$

According to Lemma 1.8, as j increases, the number d_j converges to x_0 in (2.28), and so do $h(d_j)$ and y_n.

(*Case* 3 where $c_1 < h(d_1)$ and $d_1 < h(c_1)$)

In this case, by analogy with Case 2, we see that when $n \geq ((2j+1)k+1)$, the number y_n is expressed as

$$c_1 < c_2 < \cdots < c_j < y_n < h(c_j) < \cdots < h(c_2) < h(c_1). \tag{2.29}$$

According to Lemma 1.8, as j increases, the number c_j converges to x_0 in (2.29), and so do $h(c_j)$ and y_n. □

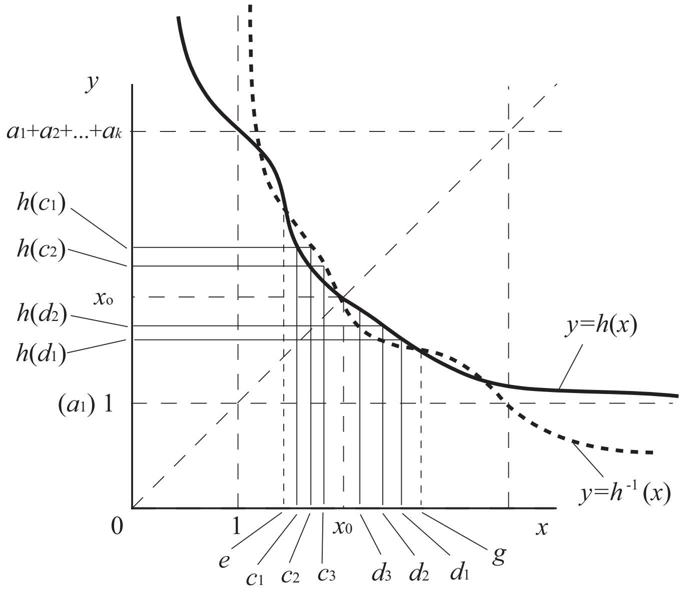

Figure 3: Another example of a non-simple function $h(x)$ and its inverse function $h^{-1}(x)$

There may be cases where even if the function $h(x)$ is not simple in the sense of Definition 1.6, the ratio $\frac{S_n}{S_{n-1}}$ converges to x_0. For example, if equation (1.11) has two pairs of real positive solutions other than x_0 as shown in Figure 3, the function $h(x)$ is not simple. In this case, we denote the two solutions nearest to x_0 as e and g, and if there exits a number n_0 such that $e < S_{n_0-i}/S_{n_0-i-1}(= y_{n_0-i}) < g$ for $i = 0, \ldots, k-1$, then we can choose c_1, d_1 such that $e < c_1 < y_{n_0-i} < d_1 < g$. Then from Figure 3, we see that as j increases, c_j and d_j converge to x_0. Thus, $\frac{S_n}{S_{n-1}}$ converges to x_0.

3 Examples

3.1 Fibonacci, Lucas, and Pell sequences

In the case where $k = 2$ and a_1 and a_2 are any natural numbers, a generalized Fibonacci-like sequence $\{S_n\}$ defined by (1.1) is expressed as

$$S_n = a_1 S_{n-1} + a_2 S_{n-2} \quad \text{for } n \geq 2, \tag{3.1}$$

where S_1 is any natural number and S_0 is a non-negative integer. The corresponding function $h(x)$ defined by (1.7) is as follows:

$$h(x) = a_1 + \frac{a_2}{x}. \tag{3.2}$$

This case includes Fibonacci numbers ($k = 2, a_1 = a_2 = 1, S_0 = 0, S_1 = 1$) [1], Lucas numbers ($k = 2, a_1 = a_2 = 1, S_0 = 2, S_1 = 1$) [3], and Pell numbers ($k = 2, a_1 = 2, a_2 = 1, S_0 = 0, S_1 = 1$) [2]. By substituting function (3.2) for $h(x)$ in equation (1.8), we obtain

$$x^2 - a_1 x - a_2 = 0. \tag{3.3}$$

The only one real positive solution x_0 to equation (3.3) is as follows:

$$x_0 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2}. \tag{3.4}$$

Corollary 3.1. *The ratio* $\frac{S_n}{S_{n-1}}$ *of the sequence defined by* (3.1) *converges to* x_0 *expressed by* (3.4).

Proof. If the function $h(x)$ defined by (3.2) is not simple, there exist two real positive numbers e, g such that

$$g = a_1 + \frac{a_2}{e} \neq e, \quad \text{and} \tag{3.5}$$

$$e = a_1 + \frac{a_2}{g} \neq x_0. \tag{3.6}$$

These equations lead to $e = g$, which means that the function $h(x)$ defined by (3.2) is simple. Thus from Theorem 2.1 we see that the ratio $\frac{S_n}{S_{n-1}}$ converges to x_0. □

3.2 A case where k is any integer

In the case where k is any integer ≥ 2 and $a_1 = a_2 = \cdots = a_k = 1$, a generalized Fibonacci-like sequence $\{S_n\}$ defined by (1.1) and the corresponding function $h(x)$ defined by (1.7) are as follows:

$$S_n = S_{n-1} + S_{n-2} + \cdots + S_{n-k} \quad \text{for } n \geq k, \tag{3.7}$$

$$\begin{aligned} h(x) &= 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^{k-1}} \\ &= \frac{(\frac{1}{x})^k - 1}{\frac{1}{x} - 1}. \end{aligned} \tag{3.8}$$

Corollary 3.2. *The ratio* $\frac{S_n}{S_{n-1}}$ *of the sequence defined by* (3.7) *converges to* x_0*, which is the real positive solution to equation* (1.8)*, where* $h(x)$ *is defined by* (3.8).

Proof. We denote function (3.8) as $h_k(x)$ and suppose that the function $h_k(x)$ is not simple. Then there exist at lease two positive real numbers e and g such that

$$g = h_k(e) = \frac{(\frac{1}{e})^k - 1}{\frac{1}{e} - 1} \neq e, \tag{3.9}$$

$$e = h_k(g) = \frac{(\frac{1}{g})^k - 1}{\frac{1}{g} - 1} \neq x_0. \tag{3.10}$$

From (3.9) and (3.10) it follows that

$$g\big(\frac{1}{e} - 1\big) = \big(\frac{1}{e}\big)^k - 1, \quad \text{thus,} \quad g\big(\frac{1}{e} - 1\big) + 1 = \big(\frac{1}{e}\big)^k, \tag{3.11}$$

$$e\big(\frac{1}{g} - 1\big) = \big(\frac{1}{g}\big)^k - 1, \quad \text{thus,} \quad e\big(\frac{1}{g} - 1\big) + 1 = \big(\frac{1}{g}\big)^k. \tag{3.12}$$

Here we multiply the both sides of the second equations of (3.11) and (3.12) by e and g, respectively. Then we obtain

$$g - eg + e = \big(\frac{1}{e}\big)^{k-1}, \tag{3.13}$$

$$e - eg + g = \big(\frac{1}{g}\big)^{k-1}. \tag{3.14}$$

Comparing (3.13) and (3.14), we see that

$$\big(\frac{1}{e}\big)^{k-1} = \big(\frac{1}{g}\big)^{k-1}, \quad \text{that is,} \quad e = g. \tag{3.15}$$

This means that the function $h_k(x)$ is simple. Thus from Theorem 2.1 it follows that the ratio $\frac{S_n}{S_{n-1}}$ converges to x_0. □

Then suppose that $k = 6$ and $S_0 = S_1 = \cdots = S_5 = 1$ in (3.7) and (3.8). In this case, equation (1.8) will be

$$x = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5}. \tag{3.16}$$

We calculated the only one real positive solution x_0 to (3.16), which was approximately 1.984. We also calculated the ratio $\frac{S_n}{S_{n-1}}$ of the sequence (3.7) for n=6,7,8,. . . . The calculation shows that the ratio converges to x_0.

Note that the ratio converges to x_0 regardless of the values of $S_0, \ldots, S_5$, provided that $S_0, \ldots, S_4$ are non-negative integers and S_5 is a natural number.

3.3 A case having a non-simple function

In the case where $k = 3$, $a_1 = a_2 = 1$, $a_3 = 10$, a generalized Fibonacci-like sequence $\{S_n\}$ defined by (1.1) and the corresponding function defined by (1.7) are as follows:

$$S_n = S_{n-1} + S_{n-2} + 10 \cdot S_{n-3} \quad \text{for } n \geq 3, \tag{3.17}$$

$$h(x) = 1 + \frac{1}{x} + \frac{10}{x^2}. \tag{3.18}$$

We denote function (3.18) as $h_3(x)$.

Lemma 3.3. *The function $h_3(x)$ is not simple in the sense of Definition* 1.6.

Proof. Suppose that there exist two real positive numbers e and g such that

$$g = h_3(e) = 1 + \frac{1}{e} + \frac{10}{e^2} > e, \tag{3.19}$$

$$e = h_3(g) = 1 + \frac{1}{g} + \frac{10}{g^2}. \tag{3.20}$$

By solving (3.19) and (3.20), we have

$$e = \frac{9 - \sqrt{41}}{2}, \quad g = \frac{9 + \sqrt{41}}{2}. \tag{3.21}$$

This means that equation (1.11), where $h(x) = h_3(x)$, has two real positive solutions e, g other than x_0. □

Thus, Theorem 2.1 does not hold for the sequence defined by (3.17). We calculated the only one real positive solution x_0 to the equation $x = h_3(x)$, which was approximately 2.720. Then, under the assumption that $S_0 = S_1 = S_2 = 1$, we also calculated the ratio $\frac{S_n}{S_{n-1}}$ of the sequence (3.17) for n=3,4,5,.... According to the calculation, as n increases the ratio seems to fluctuate about x_0, or at least converge very slowly to x_0.

4 A case where $a_1 = 0$

Let k be any integer ≥ 3, and suppose that $a_1 = 0$, $a_2 = a_3 = \cdots = a_k = 1$. Then, another Fibonacci-like sequence $\{P_n\}$ is defined by

$$P_n = P_{n-2} + P_{n-3} + \cdots + P_{n-k}, \quad \text{for } n \geq k, \tag{4.1}$$

where P_{k-1} and P_{k-2} are natural numbers and $P_i (i = 0, 1, \ldots, k-3)$ are non-negative integers. The sequence $\{P_n\}$ includes Padovan sequence ($k = 3, a_1 = 0, a_2 = a_3 = 1, P_0 = P_1 = P_2 = 1$) [4].

By substituting x for $\frac{P_{n-i}}{P_{n-1-i}} (i = 0, 1, \ldots, k-1)$ in the sequence (4.1), an equation for determining x is expressed as

$$x = \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^{k-1}}. \tag{4.2}$$

This equation can also be derived by putting $a_1 = 0$ and $a_2 = a_3 = \cdots = a_k = 1$ in equation (1.6).

Here we denote the only one real positive solution to (4.2) as x_0. It can be easily verified that $x_0 > 1$. Further we define the right-side function of (4.2) as a function $p(x)$, namely

$$p(x) = \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^{k-1}} = \frac{(\frac{1}{x})^k - \frac{1}{x}}{\frac{1}{x} - 1}. \tag{4.3}$$

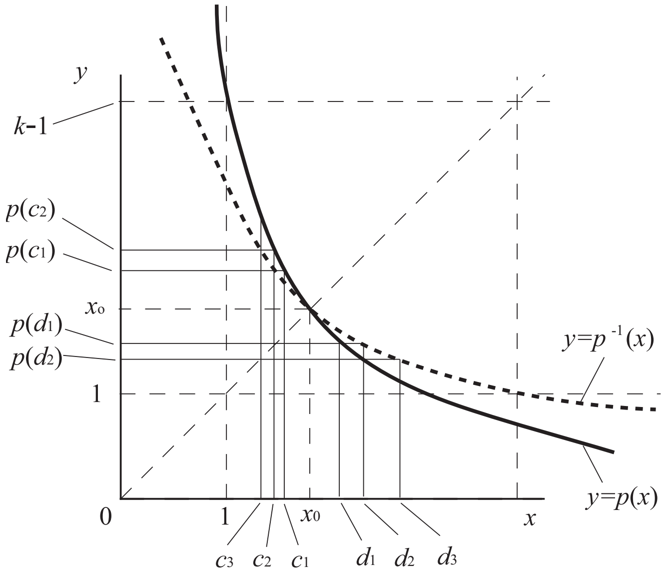

Figure 4: An example of a function $p(x)$ and its inverse function $p^{-1}(x)$

In order to check whether the function $p(x)$ is simple or not in the sense of Definition 1.6, using two real positive numbers e, g, we put

$$g = p(e) = \frac{(\frac{1}{e})^k - \frac{1}{e}}{\frac{1}{e} - 1}, \tag{4.4}$$

$$e = p(g) = \frac{(\frac{1}{g})^k - \frac{1}{g}}{\frac{1}{g} - 1}. \tag{4.5}$$

From (4.4) and (4.5) we obtain that $e = g = x_0$. Thus, since the equation $x = p(p(x))$ has no real positive solution other than x_0, the function $p(x)$ is simple.

However, in order to check whether $x < p(p(x))$ in $x < x_0$ or not, by replacing x in (4.3) by 1, we have $p(1) = k - 1$. Then, since $k \geq 3$, it follows that

$$\begin{aligned} p(p(1)) &= \frac{1}{k-1} + \frac{1}{(k-1)^2} + \cdots + \frac{1}{(k-1)^{k-1}} \\ &< \frac{1}{k-1} \times (k-1) = 1. \end{aligned} \tag{4.6}$$

This means that if $x < x_0$, then $p(p(x)) < x$ as shown in Figure 4. If we define that $c_j < x_0$, $c_{j+1} = p(p(c_j))$ and $x_0 < d_j$, $d_{j+1} = p(p(d_j))$, then from Figure 4, we see that $c_{j+1} < c_j$ and $d_j < d_{j+1}$. Thus, as j increases, c_j and d_j move away from x_0.

Hence, Lemma 1.7 and Theorem 2.1 do not hold for the sequence $\{P_n\}$ defined by (4.1).

The audience are encouraged to develop a method to deal with the sequence $\{P_n\}$.

References

[1] George E. Andrews, 1969. *Some Formulae for the Fibonacci Sequence with Generalizations*, Fibonacci Quart., 7.2: 113-130.

[2] Wayne L. McDaniel, 1996. *Triangular Numbers in the Pell Sequence*, Fibonacci Quart., 34: 105-107.

[3] Neville Robbins, 1991. *A New Formula for Lucas Numbers*, Fibonacci Quart., 29.4: 362-363.

[4] Ian Stewart, 1996. *Tales of a Neglected Number*, Sci. Am., 274: 102-103.